Christ Church
Grammar School

2020
TEST 1

MATHEMATICS SPECIALIST Year 12
Calculator-free

Your name SOLUTIONS

Teacher's name ____________________

Time and marks available for this section
Reading Time:	4 minutes
Working time for this section:	40 minutes
Marks available:	41 marks

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. The rules of conduct of the CCGS assessments are detailed in the Reporting and Assessment Policy. Sitting this assessment implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet using a blue/black pen. Do not use erasable/gel pens

3. Answer all questions.

4. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

5. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

6. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

Please turn over the find Question 1.

Question 1 **(5 marks)**

Let $z = x + yi$ be a complex number where $x > 0, y > 0$. Let $w = iz - \bar{z}$.

Determine $|w|$ in terms of x and/or y and determine the value of $\arg(w)$.

$$W = i(x+yi) - (x-yi) \checkmark$$

subst. in correctly to find w

$$= xi - y - x + yi$$

$$= -(x+y) + (x+y)i \checkmark$$

or $-x - y + (x+y)i$

or $-x - y + xi + yi$

simplifies their answer by putting real & imaginary terms together.

$$\therefore |w| = \sqrt{(-(x+y))^2 + (x+y)^2}$$

$$= \sqrt{2(x+y)^2}$$

$$= \sqrt{2}(x+y) \text{ units} \checkmark$$

correctly finds magnitude in simplified form.

$$\arg(w) = \tan^{-1}\left(\frac{x+y}{-(x+y)}\right) \checkmark$$

substs correctly into find angle

$$= \tan^{-1}(-1)$$

$$= \frac{3\pi}{4} \checkmark$$

correct value of the argument.

Note: if $\arg w = \frac{3\pi}{4}$ award 2 marks.

Question 2 **(4 marks)**

(a) Given that $a = 3\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$ and $b = 4\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$, show that $\frac{a}{b}$ is purely imaginary. (2 marks)

$a = 3\,\text{cis}\left(\frac{2\pi}{3}\right) \quad b = 4\,\text{cis}\left(\frac{\pi}{6}\right)$

$\frac{a}{b} = \frac{3}{4}\,\text{cis}\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)$

$= \frac{3}{4}\,\text{cis}\left(\frac{3\pi}{6}\right)$ ✓ correctly simplifies $\frac{a}{b}$ in polar form

$= \frac{3}{4}\,\text{cis}\left(\frac{\pi}{2}\right)$

$= \frac{3}{4}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$= \frac{3}{4}(0 + 1i)$ ✓ shows $\frac{a}{b}$ is imaginary.

$= \frac{3}{4}i \quad \therefore$ purely imaginary.

(b) Express $8\left(\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right)$ in Cartesian form. (2 marks)

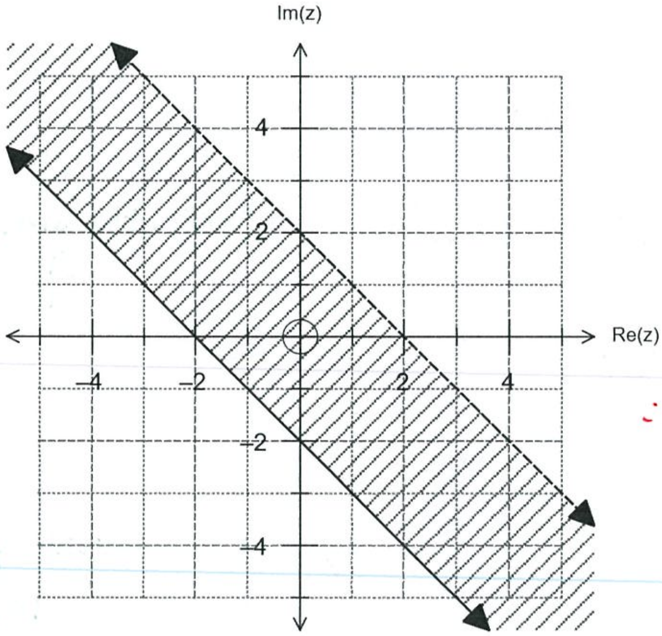
$= 8\left(\frac{1}{2} - i\cdot\frac{\sqrt{3}}{2}\right)$ ✓ correctly calculates cos & sin values

$= 4 - 4\sqrt{3}i$ ✓ correct simplified Cartesian form.

Note: correct answer only award 2marks.

Question 3 **(10 marks)**

(a) State the conditions on the complex number $z = a + bi$ that describe the region given below. (3 marks)



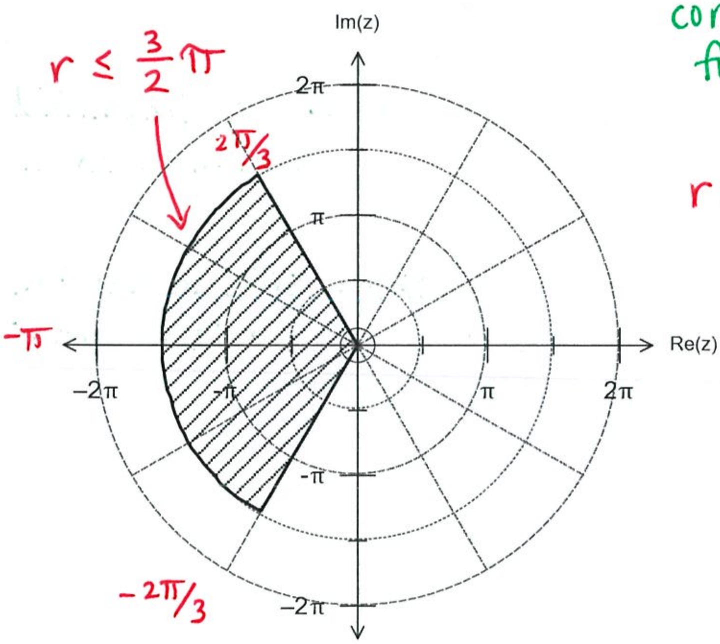
$b < -a+2$ ✓

$b \geq -a-2$ ✓

$\therefore -a-2 \leq b < -a+2$ ✓

(b) State the conditions on the complex number $z = r\,cis\,\theta$ that describe the region given below, where $r \geq 0$ and $-\pi < \theta \leq \pi$. (3 marks)

$r \leq \frac{3}{2}\pi$



correctly inequality for circle

Correct inequality with ∩ to other θ

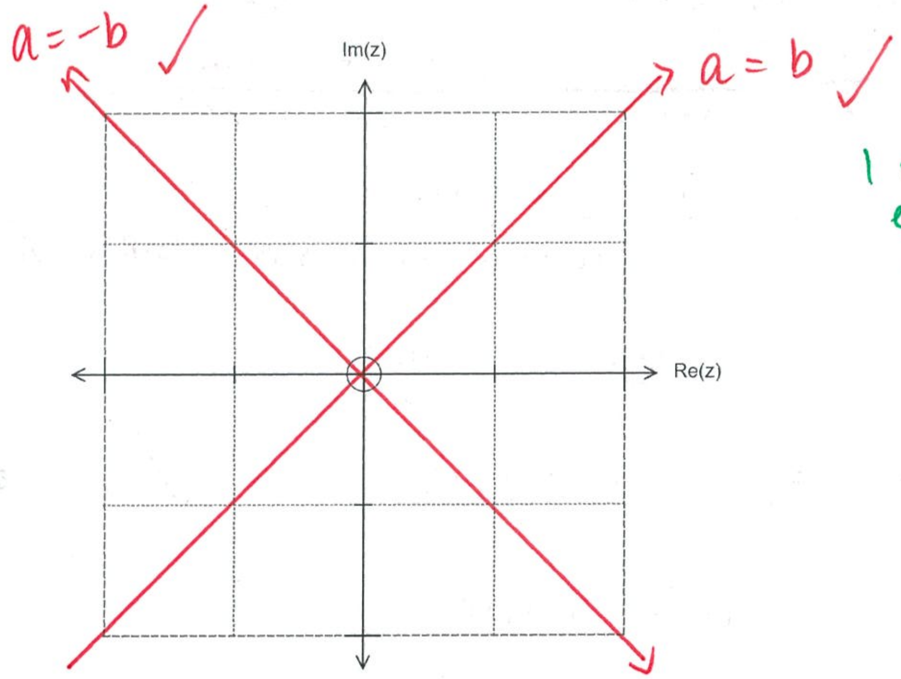
✓ ✓ ✓

$r \leq \frac{3}{2}\pi \cap -\pi < \theta \leq \frac{-2\pi}{3} \cap \frac{2\pi}{3} \leq \theta \leq \pi$

Correct θ inequality with ∩ (to circle)

Question 3 continued

(c) Sketch the following set of complex numbers z in the Argand plane that satisfy the condition: $z^2 + \bar{z}^2 = 0$. (4 marks)



1 mark for each correct line

If $z = a + bi$ then $\bar{z} = a - bi$

$\Rightarrow z^2 + \bar{z}^2 = 0$

$\Rightarrow (a+bi) + (a-bi)^2 = 0$

$\Rightarrow a^2 + 2abi - b^2 + a^2 - 2abi - b^2 = 0$

$\Rightarrow 2a^2 - 2b^2 = 0$ ✓ subst & simplify to get $2a^2 - 2b^2 = 0$

$\Rightarrow a^2 = b^2$

$a = \pm b$ ✓ Determines equations of lines

Note: Lines only on diagram $\frac{2}{4}$, must have evidence for $\frac{4}{4}$.

Question 4 **(9 marks)**

Consider the complex number $z = \cos\theta + i\sin\theta$.

(a) Rationalise $\frac{1}{z}$ to show that $z^{-1} = \cos\theta - i\sin\theta$. (2 marks)

$$\frac{1}{z} = \frac{1}{(\cos\theta + i\sin\theta)} \times \frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}$$

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \cos\theta i\sin\theta - \cos\theta i\sin\theta - i^2\sin^2\theta} \checkmark$$

multiplies denominators correctly

$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{1} \checkmark$$

simplifies toget 1 on denominator but must show identity.

$$= \cos\theta - i\sin\theta$$

(b) It can be shown that $z^n = \cos(n\theta) + i\sin(n\theta)$ and $z^{-n} = \cos(n\theta) - i\sin(n\theta)$.

Use this information to show that $\sin(n\theta) = \frac{z^n - z^{-n}}{2i}$ and $\cos(n\theta) = \frac{z^n + z^{-n}}{2}$.

(2 marks)

Real :

$$z^n + z^{-n} = \cos(n\theta) + \cos(n\theta)$$
$$= 2\cos(n\theta)$$
$$\frac{z^n + z^{-n}}{2} = \cos(n\theta) \checkmark$$

must show sum

Im:

$$z^n - z^{-n} = i\sin(n\theta) - (-i\sin(n\theta))$$
$$= 2i\sin(n\theta)$$
$$\frac{z^n - z^{-n}}{2i} = \sin(n\theta) \checkmark$$

must show sum

Question 4 continued

(c) Using your answer to (b), write an expression for $\cos(2\theta)$ and $\sin(4\theta)$ in terms of z. (2 marks)

$\cos(n\theta) = \dfrac{z^n + z^{-n}}{2}$ $\therefore \cos(2\theta) = \dfrac{z^2 + z^{-2}}{2}$ ✓

Correct answer for cos(2θ)

$\sin(n\theta) = \dfrac{z^n - z^{-n}}{2}$

$\therefore \sin(4\theta) = \dfrac{z^4 - z^{-4}}{2i}$ ✓

correct answer for ~~cos~~ sin(4θ)

(d) Using your answer to (c), determine an expression for $\cos(2\theta) \times \sin(4\theta)$ in terms of $\sin(6\theta)$ and $\sin(2\theta)$. (3 marks)

$$\cos(2\theta) \times \sin(4\theta) = \left(\frac{z^2 + z^{-2}}{2}\right)\left(\frac{z^4 - z^{-4}}{2i}\right)$$

$$= \left(\frac{z^2}{2} + \frac{z^{-2}}{2}\right)\left(\frac{z^4}{2i} - \frac{z^{-4}}{2i}\right)$$

$$= \frac{z^6}{4i} - \frac{z^{-2}}{4i} + \frac{z^2}{4i} - \frac{z^{-6}}{4i}$$ ✓

correctly expands brackets

$$= \frac{1}{2}\left(\frac{z^6 - z^{-6}}{2i} + \frac{z^2 - z^{-2}}{2i}\right)$$ ✓

correctly splits fractions and ÷ by $\frac{1}{2}$

$$= \frac{1}{2}\sin(6\theta) + \frac{1}{2}\sin(2\theta)$$ ✓

Final statement.

Question 5 **(13 marks)**

(a) The function $f(z) = 2z^3 - z^2 + 6z - 3$ is defined for $z \in \mathbb{C}$.

(i) Show that $(z + \sqrt{3}i)$ is a factor of $f(z)$. (2 marks)

then $z = -\sqrt{3}i$

$$f(-\sqrt{3}i) = 2(-\sqrt{3}i)^3 - (-\sqrt{3}i)^2 + 6(-\sqrt{3}i) - 3$$ ✓ subst $-\sqrt{3}i$ into $f(z)$

$$= -2.3\sqrt{3}i^3 - (3i^2) - 6\sqrt{3}i - 3$$

$$= +6\sqrt{3}i + 3 - 6\sqrt{3}i - 3$$ ✓ shows simplifying of subst.

$$= 0$$

(ii) Given that $(z + \sqrt{3}i)$ is a factor of $f(z)$, state another factor of $f(z)$. (1 mark)

$(z - \sqrt{3}i)$ ✓

(iii) Hence, or otherwise, solve the equation $2z^3 + 6z = z^2 + 3$ (4 marks)

$$2z^3 + 6z = z^2 + 3$$

$$\Rightarrow \quad 2z^3 + 6z - z^2 - 3 = 0$$

$$(z+\sqrt{3}i)(z-\sqrt{3}i)(az+b) = 0$$

$$(z^2+3)(az+b) = 0$$ ✓ makes statement $f(z) = (\ )(\ )$

$$(z^2+3)(2z-1) = 0$$

$\therefore az^3 = 2z^3$ $3b = -3$

$a = 2$ ✓ $b = -1$ ✓ solves for 'b'

solves for 'a'

$\therefore z = \pm\sqrt{3}i, \frac{1}{2}$ ✓ all 3 solutions stated.

Note: can get 4th mark if they solve correctly based on their $(\ )(\ )$ but must have $\pm\sqrt{3}i$ as they are given

Question 5 continued

(b) Determine all the solutions to the equation $z^3 + 2^{-3} = 0$ in the form $z = a + bi$, and then sketch all the solutions on the grid provided below. (6 marks)

$z^3 + 2^{-3} = 0$

$z^3 + \frac{1}{8} = 0$

$z^3 = -\frac{1}{8}$ ✓ re-arranges equation correctly

$z^3 = \frac{1}{8} \text{cis}\, \pi$ ✓ correctly converts to cis (polar) form

$z_1 = \sqrt[3]{\frac{1}{8}} \text{cis} \frac{\pi}{3}$ or $z_k = \frac{1}{2} \text{cis} \left(\frac{\pi + 2\pi k}{3} \right)$ for $k = 0, \pm 1$

$z_1 = \frac{1}{2} \text{cis} \frac{\pi}{3} = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{2}$
$= \frac{1}{4} + \frac{\sqrt{3}}{4} i$

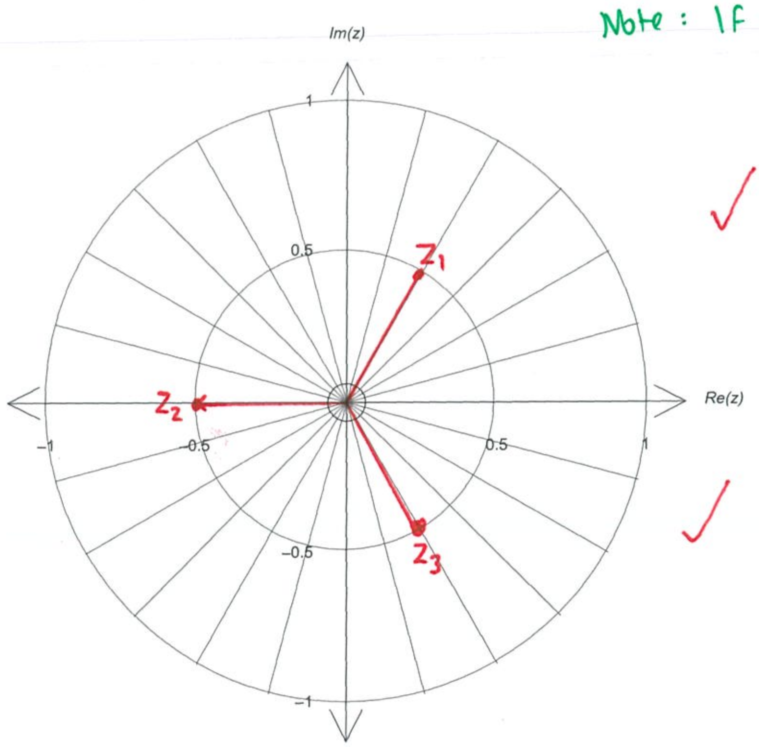
Determines a single solution correctly ✓

$z_2 = \frac{1}{2} \text{cis}\, \pi = -\frac{1}{2} + 0i$

$z_3 = \frac{1}{2} \text{cis} \left(-\pi/3 \right) = \frac{1}{4} - \frac{\sqrt{3}}{4} i$

Determines all 3 solutions in correct form ✓

Note: If solutions in polar form −1.



✓ correct modulus on all 3

✓ correct position and $\frac{2\pi}{3}$ radians apart.

End of questions

Additional working space

Question number: _________

Additional working space

Question number: __________